

TRIGONOMETRIC SERIES IN WHICH ANGLES ARE IN A.P.

Section - 5

5.1 TYPE-1

Problems based on finding the sum of series of sines or cosines whose angles are in A.P.

Following two results are very useful in solving such problems.

$$\text{Result I : } \sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + \overline{n-1} \beta) = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cdot \sin \left[\alpha + (n-1) \frac{\beta}{2} \right].$$

$$\text{Result II : } \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots + \cos (\alpha + \overline{n-1} \beta) = \frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \cdot \cos \left[\alpha + (n-1) \frac{\beta}{2} \right].$$

Proof for Result I :

Let $S = \sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots + \sin (\alpha + \overline{n-1} \beta)$

Here angles are in A.P. and common difference of angles = β

Note : In a series of sine and cosine whenever angles are in A.P. and power of sine and cosine is one then we multiply each term by $2\sin\left(\frac{\text{Common difference of angles}}{2}\right)$, then express each term as a difference of two terms and add.

So, multiplying both sides by $2\sin\frac{\beta}{2}$, we get :

$$S \cdot 2\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha + \beta) \cdot \sin\frac{\beta}{2} + \dots + 2\sin(\alpha + \overline{n-1} \beta) \sin\frac{\beta}{2} \quad \dots (i)$$

Now, first term of above R.H.S. can be written as $2\sin\alpha \cdot \sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$

Similarly, the second term R.H.S. can be written as $2\sin(\alpha + \beta) \sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$

... ..

and also the last term of R.H.S. can be written as $2\sin(\alpha + \overline{n-1} \beta) \cdot \sin\frac{\beta}{2} = \cos\left[\alpha + (n-3)\frac{\beta}{2}\right] - \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$

Now, adding above all terms, we get R.H.S. of (1) as $\cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left[\alpha + (n-1)\frac{\beta}{2}\right]$.

From (i), we get :

$$2\sin\frac{\beta}{2} \cdot S = 2\sin\left[\alpha + (n-1)\frac{\beta}{2}\right] \cdot \sin\frac{n\beta}{2} \quad \Rightarrow \quad S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Students are advised to prove the result – 2 themselves.

Illustration - 23 The value of expression $\cos 2\pi/7 + \cos 4\pi/7 + \cos 6\pi/7$ is :

- (A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) 0 (D) 1

SOLUTION : (B)

$$\begin{aligned} & \frac{2 \sin \frac{\pi}{7} \left(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} \right)}{2 \sin \frac{\pi}{7}} \\ &= \frac{1}{2 \sin \frac{\pi}{7}} \left[\left(\sin \frac{3\pi}{7} - \sin \frac{\pi}{7} \right) + \left(\sin \frac{5\pi}{7} - \sin \frac{3\pi}{7} \right) + \left(\sin \frac{7\pi}{7} - \sin \frac{5\pi}{7} \right) \right] = \frac{\sin \pi - \sin \frac{\pi}{7}}{2 \sin \frac{\pi}{7}} = -\frac{1}{2} \end{aligned}$$

Alternative Method :

We can also use the relation :

$$\cos a + \cos(a+d) + \dots + \cos(a + \overline{n-1}d) = \frac{\sin nd/2}{\sin d/2} \cos \left(\frac{2a + \overline{n-1}d}{2} \right) \quad [\text{where } d \text{ is common difference of AP}]$$

$$\begin{aligned} \Rightarrow & \frac{\sin 3 \left(\frac{2\pi/7}{2} \right)}{\sin \frac{2\pi/7}{2}} \cos \left(\frac{\frac{4\pi}{7} + 2 \left(\frac{2\pi}{7} \right)}{2} \right) \\ &= \frac{\sin \frac{3\pi}{7} \cdot \cos \frac{4\pi}{7}}{\sin \frac{\pi}{7}} = \frac{\cos \left(\frac{\pi}{2} - \frac{3\pi}{7} \right) \sin \left(\frac{\pi}{2} - \frac{4\pi}{7} \right)}{\sin \frac{\pi}{7}} = \frac{\cos \frac{\pi}{14} \sin \left(\frac{-\pi}{14} \right)}{\sin \frac{\pi}{7}} = \frac{-1}{2} \frac{\sin \frac{\pi}{7}}{\sin \frac{\pi}{7}} = -\frac{1}{2} \end{aligned}$$

5.2 TYPE-2

If angles are in A.P. and sum of the first and the last angles is π or $\pi/2$. Then we will use the following working rule and power of sine and cosine may or may not be one.

Working Rule :

If angles are in A.P. and sum of the first and last angles is $\pi/2$ or π etc., then group equidistant terms from both ends, express last angle in terms of the first in each group and simplify.

Illustration - 24 The value of expression : $\cos^4 \frac{\pi}{8} + \cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} + \cos^4 \frac{7\pi}{8}$ is :

- (A) $\frac{1}{2}$ (B) $\frac{3}{2}$ (C) $\frac{1}{4}$ (D) $\frac{3}{4}$

SOLUTION : (B)

$$\begin{aligned} & \left(\cos^4 \frac{\pi}{8} + \cos^4 \frac{7\pi}{8} \right) + \left(\cos^4 \frac{3\pi}{8} + \cos^4 \frac{5\pi}{8} \right) \\ &= \left[\cos^4 \frac{\pi}{8} + \cos^4 \left(\pi - \frac{\pi}{8} \right) \right] + \left[\cos^4 \frac{3\pi}{8} + \cos^4 \left(\pi - \frac{3\pi}{8} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= 2\cos^4 \frac{\pi}{8} + 2\cos^4 \frac{3\pi}{8} \quad [\text{As } \cos(\pi - \theta) = -\cos \theta] \\
 &= 2 \left[\left(\cos^2 \frac{\pi}{8} \right)^2 + \left(\cos^2 \frac{3\pi}{8} \right)^2 \right] = 2 \left[\left(\frac{1 + \cos \frac{\pi}{4}}{2} \right)^2 + \left(\frac{1 + \cos \frac{3\pi}{4}}{2} \right)^2 \right] \\
 &= \frac{1}{2} \left[\left(1 + \frac{1}{\sqrt{2}} \right)^2 + \left(1 - \frac{1}{\sqrt{2}} \right)^2 \right] = \frac{1}{2} \left[1 + \frac{1}{2} + 1 + \frac{1}{2} \right] = \frac{3}{2}
 \end{aligned}$$

5.3 TYPE-3

If angles are in G.P. having common ratio 2 or 1/2, terms of sine or cosine are in power one and in product.

Illustrating the Concepts :

Prove that : $\cos A \cos 2A \cos 4A \cos 8A \dots \cos 2^{n-1}A = \frac{1}{2^n \sin A} \sin(2^n A)$

Multiply above and below by $2^n \sin A$

$$\begin{aligned}
 \therefore \text{L.H.S.} &= \frac{2^{n-1}}{2^n \sin A} [2 \sin A \cos A \cos 2A \cos 4A \dots \cos 2^{n-1}A] \\
 &= \frac{2^{n-2}}{2^n \sin A} [2 \sin 2A \cos 2A \cos 4A \dots \cos 2^{n-1}A] \\
 &= \frac{2^{n-3}}{2^n \sin A} [2 \sin 4A \cos 4A \dots \cos 2^{n-1}A] \\
 &= \frac{1}{2^n \sin A} [2 \sin 2^{n-1}A \cos 2^{n-1}A] \\
 &= \frac{1}{2^n \sin A} \sin(2 \cdot 2^{n-1}A) = \frac{\sin(2^n A)}{2^n \sin A} = \text{RHS}
 \end{aligned}$$

Illustration - 25 The value of expression $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$ is :

- (A) $1/8$ (B) $1/4$ (C) $1/16$ (D) None of these

SOLUTION : (C)

In given expression : $\cos 60^\circ = \frac{1}{2}$ and $n = 3$

by using Type 3, we get :

$$\begin{aligned}
 \Rightarrow \text{L.H.S.} &= \frac{1}{2} \left[\frac{1}{2^3 \sin 20^\circ} \right] \sin(2^3 \cdot 20^\circ) \\
 &= \frac{1}{16} \times \frac{1}{\sin 20^\circ} \cdot \sin 160^\circ = \frac{1}{16 \sin 20^\circ} \cdot \sin(180^\circ - 20^\circ) \\
 &= \frac{1}{16} \left(\frac{\sin 20^\circ}{\sin 20^\circ} \right) = \frac{1}{16}
 \end{aligned}$$

Illustration - 26 The value of expression $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$ is :

- (A) $1/8$ (B) $1/16$ (C) $1/4$ (D) $3/4$

SOLUTION : (B)

By complementary rule

$$\sin \theta = \cos (90^\circ - \theta)$$

The given expression reduces to $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

which is same as above illustration.

5.4 TYPE-4

If angles are in A.P., terms in sine or cosine having power one, and in product and sum of the first and last angles is not $\pi/2$ or π etc. then

Working Rule :

- (i) Change the last angle using formula for $\pi - \theta$ etc.
- (ii) rearrange the terms in ascending order of angles.
- (iii) group the terms in two parts: One part consisting of terms having angles in G.P. and the other part consisting of remaining terms.
- (iv) Simplify

Illustration - 27 The value of expression : $\cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{6\pi}{7}$ is :

- (A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{8}$ (D) $\frac{1}{16}$

SOLUTION : (C)

$$\text{Let } y = \cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{6\pi}{7}$$

$$= \cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \left(\pi - \frac{\pi}{7} \right)$$

$$= -\cos \frac{\pi}{7} \cdot \cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7}$$

[Here there is only one group consisting of terms having angles in G.P. So, we multiply both sides by $2 \sin \pi/7$ and simplify].

$$\therefore 2y \sin \frac{\pi}{7} = - \left(2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \right) \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$$

$$\Rightarrow 2y \sin \frac{\pi}{7} = -\sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$$

$$\Rightarrow 4y \sin \frac{\pi}{7} = - \left(2 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7} \right) \cdot \cos \frac{4\pi}{7}$$

[Multiplying both side by 2]

$$\Rightarrow 4y \sin \frac{\pi}{7} = -\sin \frac{4\pi}{7} \cos \frac{4\pi}{7}$$

$$\Rightarrow 8y \sin \frac{\pi}{7} = -2 \sin \frac{4\pi}{7} \cdot \cos \frac{4\pi}{7}$$

[Multiplying both side by 2]

$$\Rightarrow 8y \sin \frac{\pi}{7} = -\sin \frac{8\pi}{7} = -\sin \left(\pi + \frac{\pi}{7} \right)$$

$$\Rightarrow 8y \sin \frac{\pi}{7} = +\sin \frac{\pi}{7} \Rightarrow y = \frac{1}{8}$$

Illustration - 28 The value of expression is $\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ - \sin^2 9^\circ - \sin^2 18^\circ$ is :

- (A) 1 (B) 0 (C) -1 (D) None

SOLUTION : (A)

$$\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ - \sin^2 9^\circ - \sin^2 18^\circ$$

$$\sin^2 12^\circ + \sin^2 21^\circ + (\sin^2 39^\circ - \sin^2 9^\circ) + (\sin^2 48^\circ - \sin^2 18^\circ)$$

Multiplying and divide by 2,

$$\frac{1}{2} [2 \sin^2 12^\circ + 2 \sin^2 21^\circ + 2 (\sin^2 39^\circ - \sin^2 9^\circ) + 2 (\sin^2 48^\circ - \sin^2 18^\circ)]$$

$$\text{By using } 2 \sin^2 \theta = 1 - \cos 2\theta \text{ and } \sin^2 \theta - \sin^2 \phi = \sin(\theta + \phi) \cdot \sin(\theta - \phi)$$

$$\Rightarrow \frac{1}{2} [1 - \cos 24^\circ + 1 - \cos 42^\circ + 2 \sin 48^\circ \sin 30^\circ + 2 \sin 66^\circ \sin 30^\circ] = 1$$

$$\text{As } \sin 30^\circ = \frac{1}{2} \text{ and by complementary rule } \sin 66^\circ = \cos 24^\circ; \sin 48^\circ = \cos 42^\circ.$$

Illustration - 29 The value of expression : $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ$ is :

- (A) $\frac{3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{16}$ (D) $\frac{1}{2}$

SOLUTION : (C)

Note that $(66 + 6)/2 = 36$ and $(66 - 6)/2 = 30$. Hence $\sin 6^\circ$ and $\sin 66^\circ$ should be combined.

$$= \frac{1}{4} [2 \sin 6^\circ \sin 66^\circ] [2 \sin 42^\circ \sin 78^\circ]$$

$$= \frac{1}{4} [\cos(6^\circ - 66^\circ) - \cos(6^\circ + 66^\circ)] [\cos(42^\circ - 78^\circ) - \cos(42^\circ + 78^\circ)]$$

$$= \frac{1}{4} [\cos 60^\circ - \cos 72^\circ] [\cos 36^\circ - \cos 120^\circ]$$

Substituting the values, we get :

$$\begin{aligned} &= \frac{1}{4} \left(\frac{1}{2} - \frac{\sqrt{5}-1}{4} \right) \left(\frac{\sqrt{5}+1}{4} + \frac{1}{2} \right) = \frac{1}{4} \left(\frac{2-\sqrt{5}+1}{4} \right) \left(\frac{\sqrt{5}+1+2}{4} \right) \\ &= \frac{1}{64} (3-\sqrt{5})(3+\sqrt{5}) = \frac{1}{16} = \text{R.H.S.} \end{aligned}$$

Illustration - 30 The value of expression : $\sin 20^\circ \sin 40^\circ \sin 80^\circ$ is :

- (A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{16}$ (D) $\frac{\sqrt{3}}{8}$

SOLUTION : (D)

Consider $\sin 20^\circ \sin 40^\circ \sin 80^\circ$

$$= \sin 20^\circ \sin (60^\circ - 20^\circ) \sin (60^\circ + 20^\circ)$$

$$= \frac{1}{4} \sin 3(20^\circ) = \frac{1}{4} \sin 60^\circ = \frac{\sqrt{3}}{8}$$

Illustration - 31

The value of expression : $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14}$ is :

- (A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{16}$

SOLUTION : (A)

$$(i) \quad \sin \frac{\pi}{14} = \sin \left(\frac{\pi}{2} - \frac{6\pi}{14} \right) = \cos \frac{6\pi}{14} = \cos \left(\pi - \frac{8\pi}{14} \right) = -\cos \frac{8\pi}{14}$$

$$\sin \frac{3\pi}{14} = \sin \left(\frac{\pi}{2} - \frac{4\pi}{14} \right) = \cos \frac{4\pi}{14}$$

$$\sin \frac{5\pi}{14} = \sin \left(\frac{\pi}{2} - \frac{2\pi}{14} \right) = \cos \frac{2\pi}{14}$$

$$\begin{aligned} \therefore \text{L.H.S.} &= -\cos \frac{2\pi}{14} \cos \frac{4\pi}{14} \cos \frac{8\pi}{14} \\ &= -\frac{1}{2^3 \sin A} \cdot \sin (2^3 A) \quad \left[\text{where } A = \frac{2\pi}{14} \right] \\ &= -\frac{1}{8 \sin \frac{\pi}{7}} \sin \frac{8\pi}{7} = -\frac{1}{8 \sin \frac{\pi}{7}} \sin \left(\pi + \frac{\pi}{7} \right) = -\frac{1}{8} (-1) = \frac{1}{8} \quad [\sin (\pi + \theta) = -\sin \theta] \end{aligned}$$

Illustration - 32

The value of expression : $\left(1 + \cos \frac{\pi}{10} \right) \left(1 + \cos \frac{3\pi}{10} \right) \left(1 + \cos \frac{7\pi}{10} \right) \left(1 + \cos \frac{9\pi}{10} \right)$ is :

- (A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{16}$ (D) $\frac{\sqrt{3}}{8}$

SOLUTION : (C)

By supplementary rule : $\cos (\pi - A) = -\cos A$

$$\begin{aligned} \text{L.H.S.} &= \left(1 + \cos \frac{\pi}{10} \right) \left(1 + \cos \frac{3\pi}{10} \right) \left(1 - \cos \frac{3\pi}{10} \right) \left(1 - \cos \frac{\pi}{10} \right) \\ &= \left(1 - \cos^2 \frac{\pi}{10} \right) \left(1 - \cos^2 \frac{3\pi}{10} \right) = \sin^2 18^\circ \sin^2 54^\circ = \left(\frac{\sqrt{5}-1}{4} \cdot \frac{\sqrt{5}+1}{4} \right)^2 = \frac{1}{16} \end{aligned}$$

Illustration - 33

The value of expression : $\cos 60^\circ \cos 36^\circ \cos 42^\circ \cos 78^\circ$ is :

- (A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{1}{16}$ (D) $\frac{\sqrt{3}}{8}$

SOLUTION : (C)

Consider L.H.S. :

$$\begin{aligned}
 &= \frac{1}{2} \cdot \frac{\sqrt{5}+1}{4} \cdot \frac{1}{2} \cdot (2 \cos 42^\circ \cos 78^\circ) = \frac{1}{16} (\sqrt{5}+1) (\cos 120^\circ + \cos 36^\circ) \\
 &= \frac{1}{16} (\sqrt{5}+1) \left(-\frac{1}{2} + \frac{\sqrt{5}+1}{4} \right) = \frac{1}{16} (\sqrt{5}+1) \cdot \frac{\sqrt{5}-1}{4} = \frac{1}{16} \cdot \frac{5-1}{4} = \frac{1}{16}.
 \end{aligned}$$

Illustration - 34

For a positive integer n , let $f_n(\theta) = \left(\tan \frac{\theta}{2} \right) (1 + \sec \theta) (1 + \sec 2\theta) (1 + \sec 4\theta) \dots (1 + \sec 2^n \theta)$. Then :

(A) $f_2\left(\frac{\pi}{16}\right) = 1$ (B) $f_3\left(\frac{\pi}{32}\right) = 1$ (C) $f_4\left(\frac{\pi}{64}\right) = 1$ (D) $f_5\left(\frac{\pi}{128}\right) = 1$

SOLUTION : (ABCD)

We have $\left(\tan \frac{\theta}{2} \right) (1 + \sec \theta) = \frac{\sin(\theta/2)}{\cos(\theta/2)} \left[1 + \frac{1}{\cos \theta} \right] = \frac{\sin(\theta/2)}{\cos(\theta/2)} \cdot \frac{2 \cos^2(\theta/2)}{\cos \theta} = \frac{2 \sin(\theta/2) \cos(\theta/2)}{\cos \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$

Thus, $f_n(\theta) = \left(\tan \frac{\theta}{2} \right) (1 + \sec \theta) (1 + \sec 2\theta) (1 + \sec 4\theta) \dots (1 + \sec 2^n \theta)$

$$\begin{aligned}
 &= (\tan \theta) (1 + \sec 2\theta) (1 + \sec 4\theta) \dots (1 + \sec 2^n \theta) \\
 &= (\tan 2\theta) (1 + \sec 4\theta) \dots (1 + \sec 2^n \theta) = (\tan 4\theta) (1 + \sec 8\theta) \dots (1 + \sec 2^n \theta) \\
 &= \dots \\
 &= \tan(2^n \theta)
 \end{aligned}$$

Now, $f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2 \frac{\pi}{16}\right) = \tan\left(\frac{\pi}{4}\right) = 1$, $f_3\left(\frac{\pi}{32}\right) = \tan\left(2^3 \frac{\pi}{32}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f_4\left(\frac{\pi}{64}\right) = \tan\left(2^4 \frac{\pi}{64}\right) = \tan\left(\frac{\pi}{4}\right) = 1$ and $f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5 \frac{\pi}{128}\right) = \tan\left(\frac{\pi}{4}\right) = 1$.